Knot

1. The Knot Problem

We will focus on the simplest case of classification of embeddings in the elementary form here. It involves embedding of the 1-sphere S^1 (circle) into the 3 dimensional Euclidean space $\boldsymbol{R}^3$.

To work in the domain of more complicated spaces, we can make $\boldsymbol{R}^3$ into the compact space 3-sphere S^3 and obtain an equivalent problem. In order to proceed in the approach, we set this problem on *tame* embeddings, which are simplicial and result in more appropriate trangulations of S^3 and S^1. Thus, we exclude *wild* embeddings, which are knotted endlessly in a certain manner, although there are also interesting examples in such cases. It should be mentioned that, according to the Jordan-Schönflies theorem, every two embeddings from S^1 into S^2 are topologically equivalent in this relation. On the other hand, there are wild embeddings from S^2 into S^3 (Antoine 1921, Alexander 1924), while every two tame (embeddings) are even combinatorially equivalent (Alexander 1924, Graeub 1950).

It has been proved appropriate to consider *orientations*. Therefore, we considered tame embeddings of the orientations on S^1 into the orientations on S^3.

Different ways are provided for the *classification*, for example

1. topological transformations of S^3,
2. strong isotopes,
3. piecewise linear (semilinear) topological transformations of S^3,

4. elementary combinatorial equivalence according to Reidemeister (Knot Theory 1932).

The transformations under 1 and 3 should maintain the orientations of the S^3. Strong isotopies are deformations of identity maps of S^3. Weak isotopies are deformations of embeddings. Every two embeddings with weak isotopies are equivalent.

The different ways of classification above are equivalent in the case we are considering. The equivalence of 1 and 2 results from that every topologically orientation-preserving self-map of the S^3 is isotopic to the identity (G.M. Fisher 1960). Graeub showed the equivalence of 3 and 4 in 1950, by which we can further attain the transition of elementary deformations to a transformation group. The equivalence of 1 and 3 is based on the fact that 3 dimensional manifolds are triangulable and that the so-called Hauptvermutung for them is valid, so then every two triangulations of the same manifold have isomorphic partitions (Moise 1952, 1954, Bing 1959). In addition, another classification is clear: one can consider differentiable embeddings and define equivalence with respect to diffeomorphisms of the S^3. Also, this is proved to be equivalent with the ones that are previously mentioned.

It is remarkable that this equivalence does not exist if we consider generic embeddings from S^r into S^n. Here in addition, we require that the tame embeddings are locally flat, which means that every point on the embedded S^r has a neighborhood in S^r. The neighborhood can be made through a topological or piecewise linear self-map of the S^n. This comes automatically with the case of differentiable embeddings. However, there are combinatorial embeddings from S^2 into S^4 that are not locally flat. We can obtain the simplest case from a knotted S^1 in $\boldsymbol{R^3} \subset \boldsymbol{R^4}$ through an image of a double-cone (suspension) in $\boldsymbol{R^4}$. The difference between the combinatorial case and the differential case is even deeper. According to Zeeman (1960), for $n - r \geq 3$, every two combinatorial embeddings from S^r into S^n are equivalent with respect to piecewise linear maps. However, Haefliger showed in 1962 that the (4k-1)-sphere can be embedded (with standard differential structure of the spheres) into the 6k-sphere differentiably knottedly (thus it is not equivalent to a large sphere). The classes of differentiable knots form a group in combinatorially trivial domain $n - r \geq 3$(Haefliger 1962, Levine 1965). Although the hereby mentioned list of questions here attracted vigorous interest, led to numerous important results in recent years and provided interesting open problems, we want to limit ourselves here again to the initial case.

2. Direct Geometric Approach

We henceforth consider with the piecewise linear perspective, which indicates no loss of generality according to the statements above. Thus, a knot is now a class of equivalent and oriented simple closed polygons in the oriented S^3, whereby a triangulation of the S^3 is taken as a basis together with its linear subdivisions. The class represented by the boundary of a triangle is called circle. All the others are truly knotted.

The simplest way to represent a knot is by a regular planar projection (in $\boldsymbol{R^3}$). This means it only contains simple double points. Fig. 1 up to Fig. 4 show some examples. Fig. 1 represents the circle, Fig. 2 and 3 represent a right and a left-handed trefoil respectively, and Fig. 4 represents the four-knot. The *genus* has been proven to be a particularly important invariant (Seifert 1934). Each knot curve borders orientable (simply punctured) surfaces. The smallest genus of such surfaces is the genus of the knot. The circle is characterized by genus 0.

With the effort to classify knots, people found it interesting to investigate the methods to reduce knots to “simpler” ones, or conversely, to make new and “more complicated” ones from given knots. The oldest such method is the *product* of knots, which can be explained in the simplest way as such: we consider two knot curves in $\boldsymbol{R^3}$ that are located on different sides of a face, slide them together so that they coincide with different orientations in a line segment, and then remove the line segment (Fig. 5). This product is associative and commutative, and the circle is the identity. In the process, there exists a unique prime factorization (Schubert 1949).

Another method is the formation of *hose knots* (also called parallel knots). We thicken a knot curve *k* to a full ring and consider a knot curve on its boundary, which at least circulates twice. *k* is the support of the hose knot created in this way. Fig. 2, 3, 6 are examples with the circle as support. Hose knots can be classified by support and two numerical invariants (winding and linking number) (Schubert 1953).

Furthermore, one method is the formation of *noose*. We twine and twist a ribbon and hang the ends onto each other, as the upper portion of Fig. 4 shows. The boundary of the ribbon then forms a noose. Such knots can be classified by the transverse knot, which results from the middle line of the ribbon, and by two numerical invariants (Seifert 1949).

The methods mentioned (so far) led to the concept of the *companion knots*. Given a knot represented by a knot curve *k*, a companion knot (assuming one exists) is a purely knotted full ring *V* that contains *k* inside where *k* is neither a core (graphic middle line) nor a (topological) entire sphere in *V*.

These observations took a definitive turn due to Haken, who stated a finite procedure (published in 1961) to define the genus of a knot. What is remarkable about this is that it can be decided whether a presented knot is a circle. Finite (even if functionally infeasible due to its length) procedures for prime factorization of knots (Schubert 1961), for realizing and determining the invariant of the hose knot (Haken, unpublished so far) and of nooses with the exception of the four-knot (Soltsien 1965) and for identifying all companion knots (Hammer 1963) arise from this method. It can be proved that knots with companion knots giving clampable orientable surfaces of minimal genus are divided into finite many isotopy classes (Schubert and Soltsien 1964).

Haken's method also provides a decomposition of compact 3 dimensional manifolds into irreducible components (Haken 1961). This opens up perspectives to make the equivalence problem for knots decidable through decomposition of the exterior spaces.

3. Knot and Exterior Space

To be able to consider the exterior space of a knot as a compact manifold (with boundary), let a regular open neighborhood of a knot curve be removed from the S^3. The knot itself is then described as a curve circulating once on the boundary of a torus, which is homologous to zero in the exterior space (parallel of latitude or longitude). In addition, a conjugate loop-cut on the boundary of a torus is homologous to zero on the removed full ring (meridian).

Two fundamental results by Papakyriakopoulos should be mentioned here beforehand: the sphere theorem and the proof of Dehn’s lemma (1957), which was subsequently generalized to the loop theorem (Shapiro and J.H.C. Whitehead 1958, Stallings 1960). The *sphere theorem* says: If the second homotopy group of a triangulated 3-manifold is not trivial, then there is no null-homotopic simplicial 2-sphere in it. The *loop theorem* states: If a curve u lies on the boundary $\dot{M}$ of a 3-manifold M, which is null-homotopic in M but not in $\dot{M}$, then there is a simple and closed curve v of u on $\dot{M}$ in an arbitrarily small neighborhood, which is not null-homotopic on $\dot{M}$ and borders a simplicial 2-cell (“disk”, elementary surface) in M. Both theorems can be refined.

It follows from the sphere theorem that the second homotopy group of the exterior space of a knot vanishes. We can recognize that the higher homotopy groups will also vanish if we apply Hurewicz isomorphism theorem on the universal superposition of a 2-dimensional deformation retract. So, only the fundamental group remains, which is called the *knot group* here in most cases. Generators and relations from the projection of a knot can be indicated for this group.

The first homology group does not contribute to the differentiation of exterior spaces of knots at the beginning. It is always cyclical freely. However, it allows us to consider cyclic superpositions and their homologies simultaneously for all exterior spaces of knots, by which we obtain invariants, which are simple to calculate (in particular Seifert 1934, Threlfall 1949). The first homology group of the infinite cyclic superposition can therefore be understood as a module over the ring group of the deck transformation group. We obtain the corresponding information of this module from generators and relations of the knot group. Then, we can order generated ideals (fitting ideals) of the coefficient rings for it (Alexander 1928, Fox 1953, 1954, Crowell 1960, 1961, 1963, Gamst 1967). However, all named homologous invariants are very weak. Seifert showed in 1950 that in each case, infinitely many knots can match in these invariants.

We now turn to questions about what stems from the orientations of S^3 and S^1. The homology groups of the finite-pedal cyclic superpositions of the exterior space of knots are groups with convolution (Seifert 1935). By associated invariants, it can be shown in special cases that a knot is different from its mirror reflection. Therefore, the formerly used classes of quadratic forms and their Minkowski invariants are defined by the convolution invariants of the two-pedal superpositions (Kneser and Puppe 1953). All torus knots (hose knots whose support is a circle) are different from their mirror reflections. This could be shown by looking at the automorphism groups of the associated knot groups (Dehn 1914, Schreier 1924). Therefore, Fig. 2 and 3 represent different knots. All torus knots and nooses go over into themselves if they are reoriented. In 1964, Trotter finally successfully proved that certain knots with reorientation do not pass over themselves by showing a class of such knots. The strategy used in the process was the peripheral structure of the knot group, which was introduced by Fox in 1952. It involves a class of conjugate subgroups with perfect generators, which arise from parallels of latitude and meridian. With these ideas, many knots can also be distinguished by their mirror reflections or reoriented mirror reflections, and exterior spaces of knots with isomorphic fundamental groups can be recognized as not homeomorphic (Fox 1952).

We just said that the fundamental group does not determine the type of homeomorphism of the exterior space of knots in general. However, this is true with the circle, which follows from Dehn's lemma. Dehn had formulated the lemma for it in 1910. This is also true for torus knots. These are yet the only knots whose group has a center. This conjecture by Neuwirth was proved simultaneously for multiple times in 1965 (Burde and Zieschang, Waldhausen, Noga). The conjecture that is unproved so far comes from Fox. It states that group and peripheral subgroups characterize the exterior space.

Here, the question concludes by whether or not the oriented exterior space determines the knot except for its orientation. This is assumed to be true. However, it is only known for the circle, for the torus knots and recently for the product knots (Noga 1967) so far. An analogous statement for links (embeddings of a disjoint sum of 1-spheres) is false (J.H.C. Whitehead 1937).

The commutator group of the knot group, by Neuwirth in particular, attracted interest in recent years (reported in Ann. Math. Studies 56, Princeton 1965). By the loop theorem, we can yield that if it is free, then it is precisely finitely generated. This is equivalent to the fact that the exterior space of a knot has a Stallings' fiberation (Stallings 1961). Here, the fiber is an orientable surface with minimal genus, which is bordered by the knot, and the basis is a circular line. The four-knot and all torus knots are (such) examples. With this, the characterization of the groups and the exterior spaces of torus knots mentioned above could be attained. Then, it follows that Stallings' fiberation leads to Seifert fibration with exceptional fibers (Seifert 1933). By Haken's methods, we can determine whether or not a given knot is a Neuwirth knot, which is to say that its group has a free commutator group.

By the end, it should be mentioned explicitly that no comprehensiveness could be achieved by one report in this setting. The selection of work is limited to personal preferences. We further refer to: Crowell, R.H. and Fox, R.H.: *Introduction to Knot Theory*, Boston, 1963, and the given *Guide to the Literature* there.