Efficient Learning of Models for Temporal Networks

Moyi Tian*†, Abhijith Jayakumar†, Marc Vuffray† and Andrey Lokhov†

∗ Division of Applied Mathematics, Brown University † Theoretical Division, Los Alamos National Laboratory

Background: Exponential Random Graph Models

- **Exponential Random Graph Models (ERGMs)** is a family of statistical models popularly used in social and biological science problems, which often involve analyzing populations of interrelated agents. This family of models efficiently describes the network structures of interest specified by sufficient statistics, reflecting uncertainty of observations and permitting statistical inferences.
  Consider a graph with set of nodes V. Let $\mathbf{e} = (e_{ij})_{i,j\in V}$ be the vector of edges where $e_{ij} \in \{0,1\}$. The **general form of ERGMs** is
  $$\mathbb{P}(\mathbf{e}) = \frac{1}{Z}\exp\left\{\sum_{c\in\mathcal{C}}\theta_c f_c(\mathbf{e}_c)\right\}$$
  where Z is the normalization constant, θ_c is the parameter corresponding to factor c, and f_c is the basis function on $\mathbf{e}_c \subseteq \mathbf{e}$ defining the network statistics.
- **Temporal Exponential Random Graph Models (TERGMs)**, incorporating the Markov property, was proposed as an extension to ERGMs that further captures the dynamic evolution of network topologies over time:
  $$\mathbb{P}\left(\mathbf{e}^{t+1} \mid \mathbf{e}^t\right) = \frac{1}{Z}\exp\left\{\sum_{c\in\mathcal{C}}\theta_c f_c(\mathbf{e}_c^t, \mathbf{e}_c^{t+1})\right\}$$
  An interesting special case for studies: $\mathbb{P}\left(\mathbf{e}^{t+1} \mid \mathbf{e}^t\right) = \frac{1}{Z}\exp\left\{\sum_{c\in\mathcal{C}}\theta_c\left(\sum_{e_{ij}\in c} f_{ij,c}(\mathbf{e}_c^t, e_{ij}^{t+1})\right)\right\}$
- We propose **Exponential Random Hypergraph Models (ERHGMs)** and **Temporal Exponential Random Hypergraph Models (TERHGMs)** as extensions of ERGMs and TERGMs to hypergraphs where we consider random variables to be hyperedges $\mathbf{e} = (e_u)_{u\subseteq V}$. It is crucial to introduce **hypergraphs** to ERGMs since simple graphs cannot capture certain network relations or features, e.g., collaborations or communities.
- We present an efficient learning framework for parameters reconstruction applicable to these random graph models.

Learning of (T)ER(H)GMs

Learning algorithm based on mapping to discrete graphical models [2] with rigorous guarantees on the parameters reconstruction:

Suppose we have M i.i.d. samples $\mathbf{e}^{(1)}, \mathbf{e}^{(2)}, \ldots, \mathbf{e}^{(M)}$ from $\mathbb{P}(\mathbf{e})$.

Define locally centered basis functions for random variable e_{ij} with respect to factor c: $g_{ij,c}(\mathbf{e}_c) = f_c(\mathbf{e}_c) - \frac{1}{2}\sum_{e_{ij}\in\{0,1\}} f_c(\mathbf{e}_c)$

The idea for temporal and hypergraph cases follows the same template.

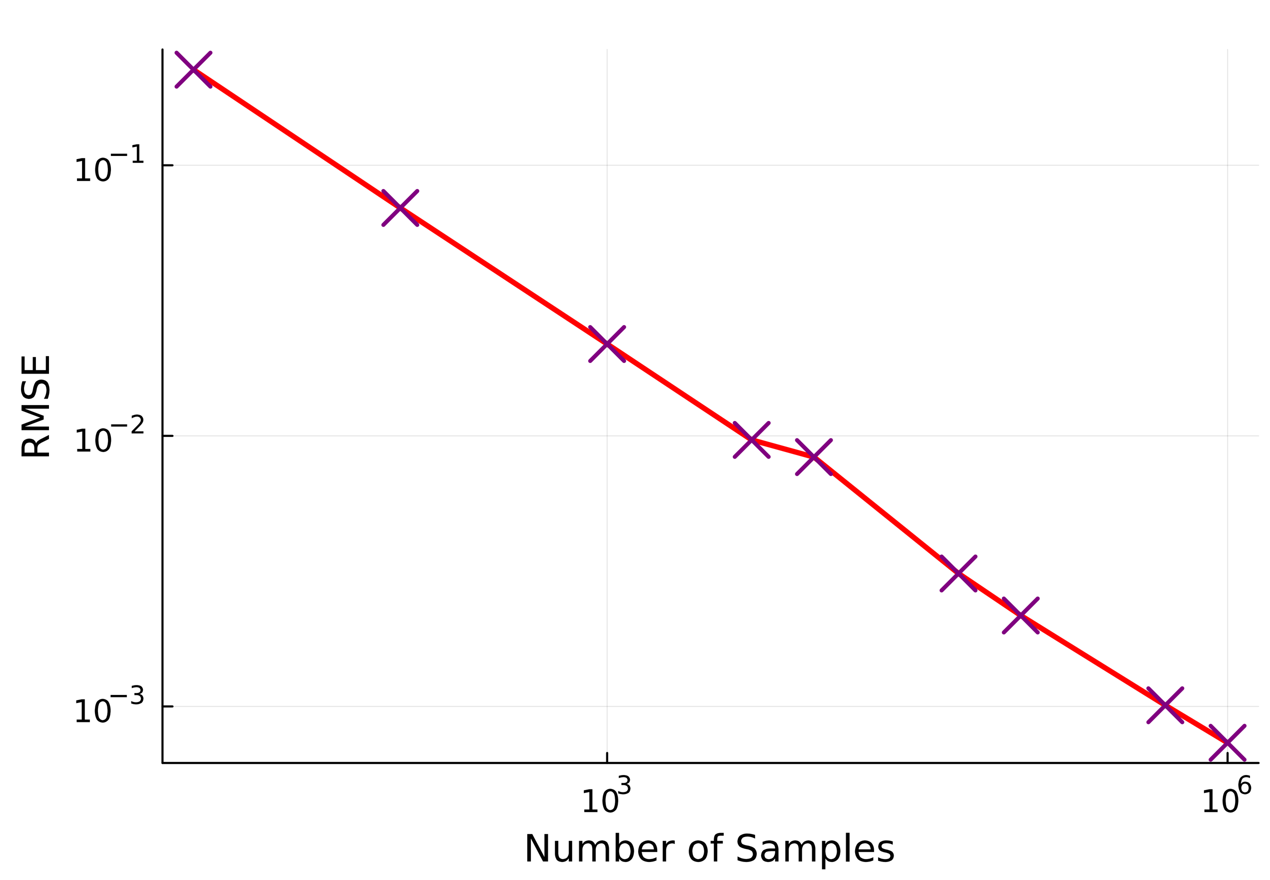
[**Convex objective function** that is easy to optimize]
Generalized interaction screening objective for $\theta_{\mathbf{ij}}$, the vector of parameters associated with factors containing e_{ij} and functions $g_{ij,c}$: $\mathcal{S}_M(\theta_{\mathbf{ij}}) = \frac{1}{M}\sum_{m=1}^{M}\exp\left\{-\sum_c \theta_c g_{ij,c}\left(\mathbf{e}_c^{(m)}\right)\right\}$

[**Estimator that is consistent and nearly sample-optimal** in case when bounds are known]
Generalized regularized interaction screening estimator for parameters $\theta_{\mathbf{ij}}$: $\hat{\theta}_{\mathbf{ij}} = \operatorname{argmin}_{\theta_{\mathbf{ij}}}\mathcal{S}_M(\theta_{\mathbf{ij}})$

Estimator for θ_c can be computed as $\hat{\theta}_c = \frac{1}{|c|}\sum_{e_{ij}\in c}\hat{\theta}_{\mathbf{ij},c}$

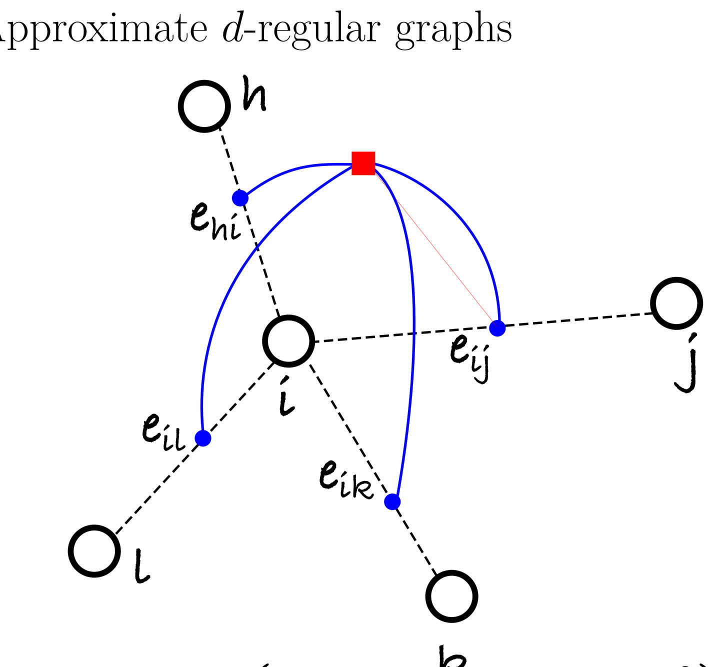
Examples

ERGM



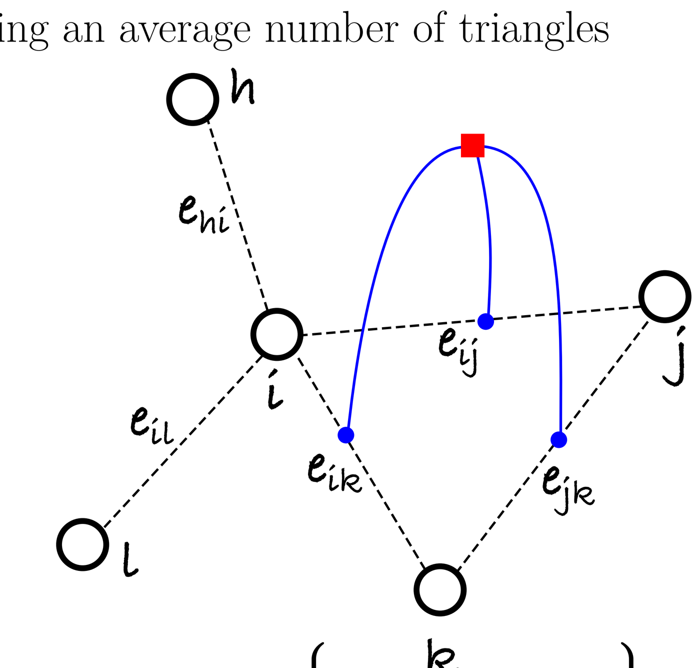
Root mean square error inversely proportional to square root of sample size (log-log plot for 5-node graphs corresponding to model (2), 100 runs each)

(1) Approximate d-regular graphs



$$\mathbb{P} = \frac{1}{Z}\exp\left\{\theta\sum_{i\in V}\left(\sum_{j\in V\setminus\{i\}} e_{ij} - d\right)^2\right\}$$

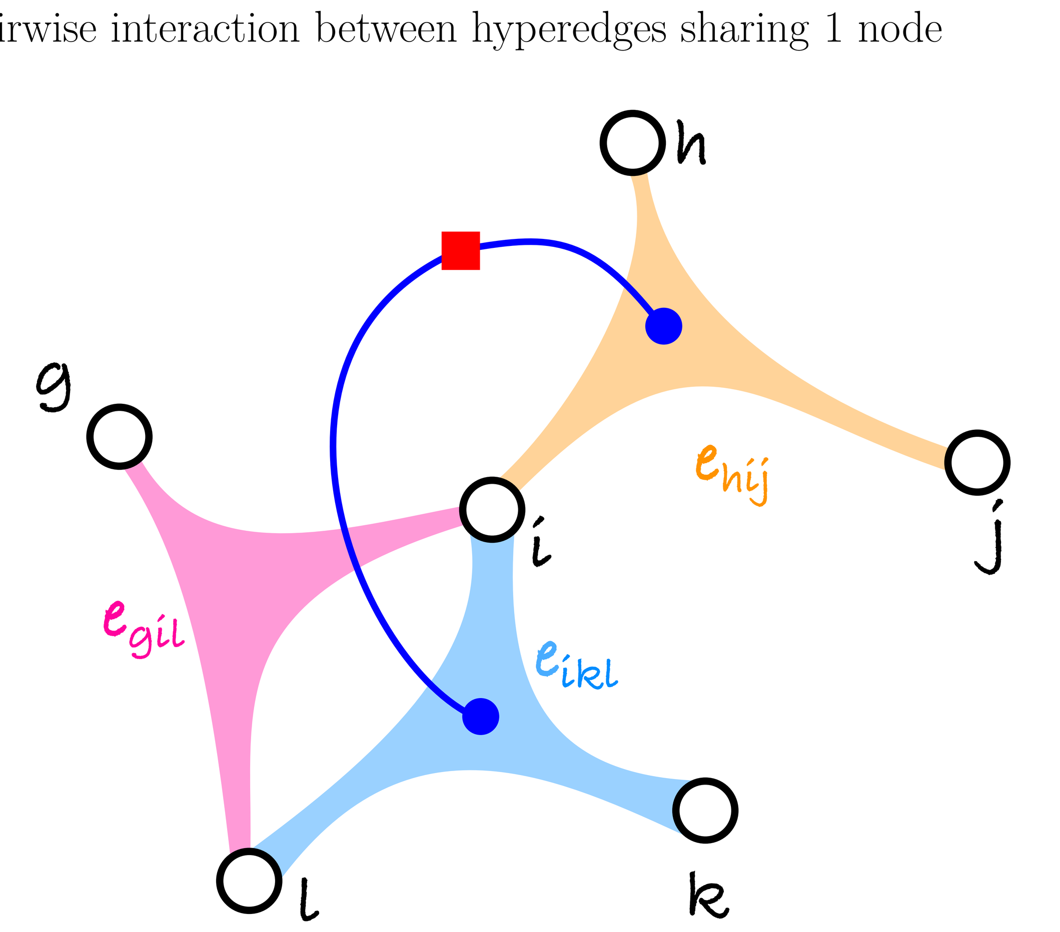
(2) Fixing an average number of triangles



$$\mathbb{P} = \frac{1}{Z}\exp\left\{\theta\sum_{\substack{i<j<k\\ i,j,k\in V}} e_{ij}e_{jk}e_{ik}\right\}$$

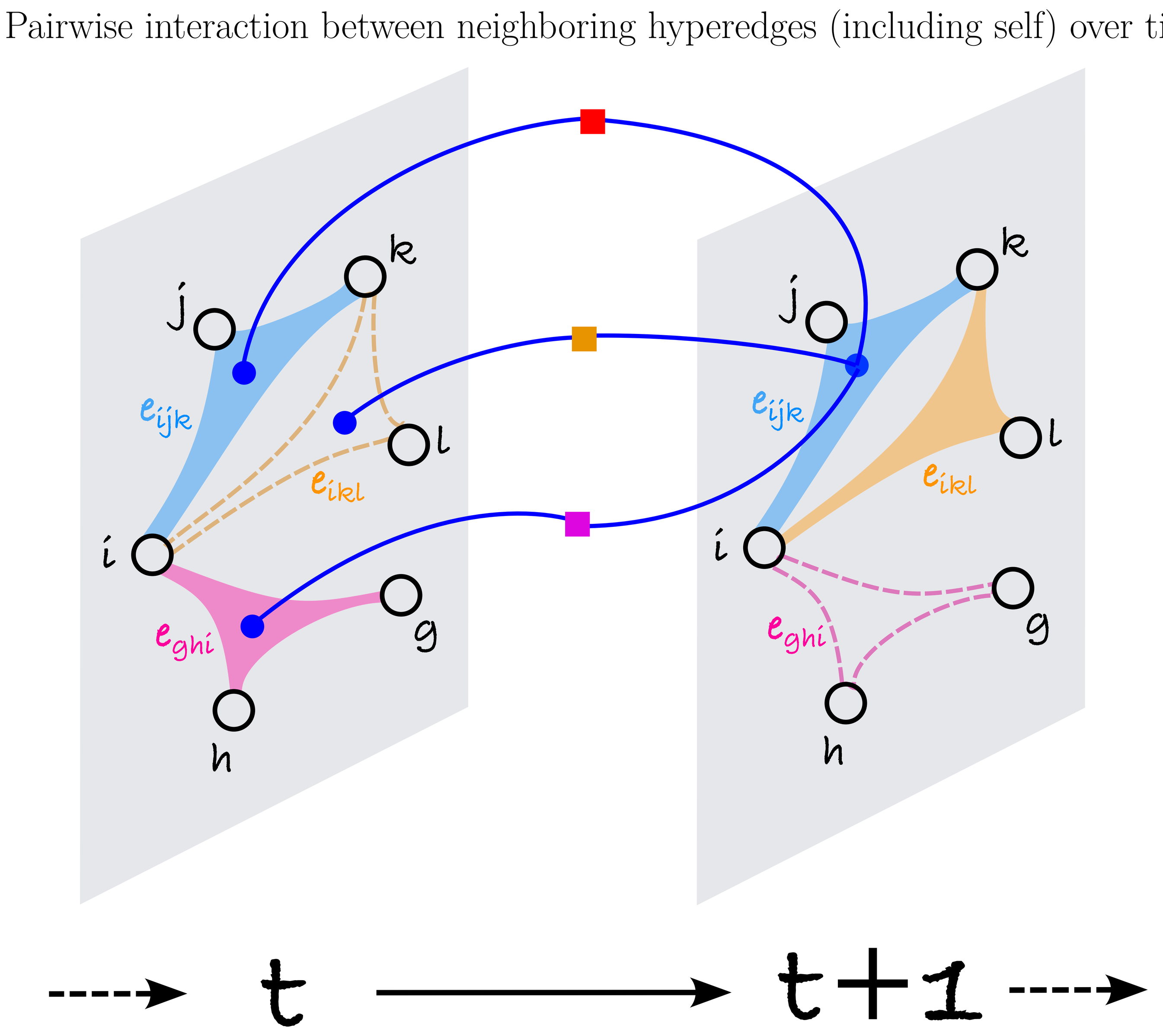
ERHGM

Pairwise interaction between hyperedges sharing 1 node



TERHGM

Pairwise interaction between neighboring hyperedges (including self) over time



References

[1] M. Tian, A. Jayakumar, M. Vuffray and A. Lokhov. In prep.

[2] M. Vuffray, S. Misra and A. Lokhov. Efficient Learning of Discrete Graphical Models. *In Proceedings of the 34th International Conference on Neural Information Processing Systems (NIPS'20)*, 1139, pages 13575–13585, (2020).