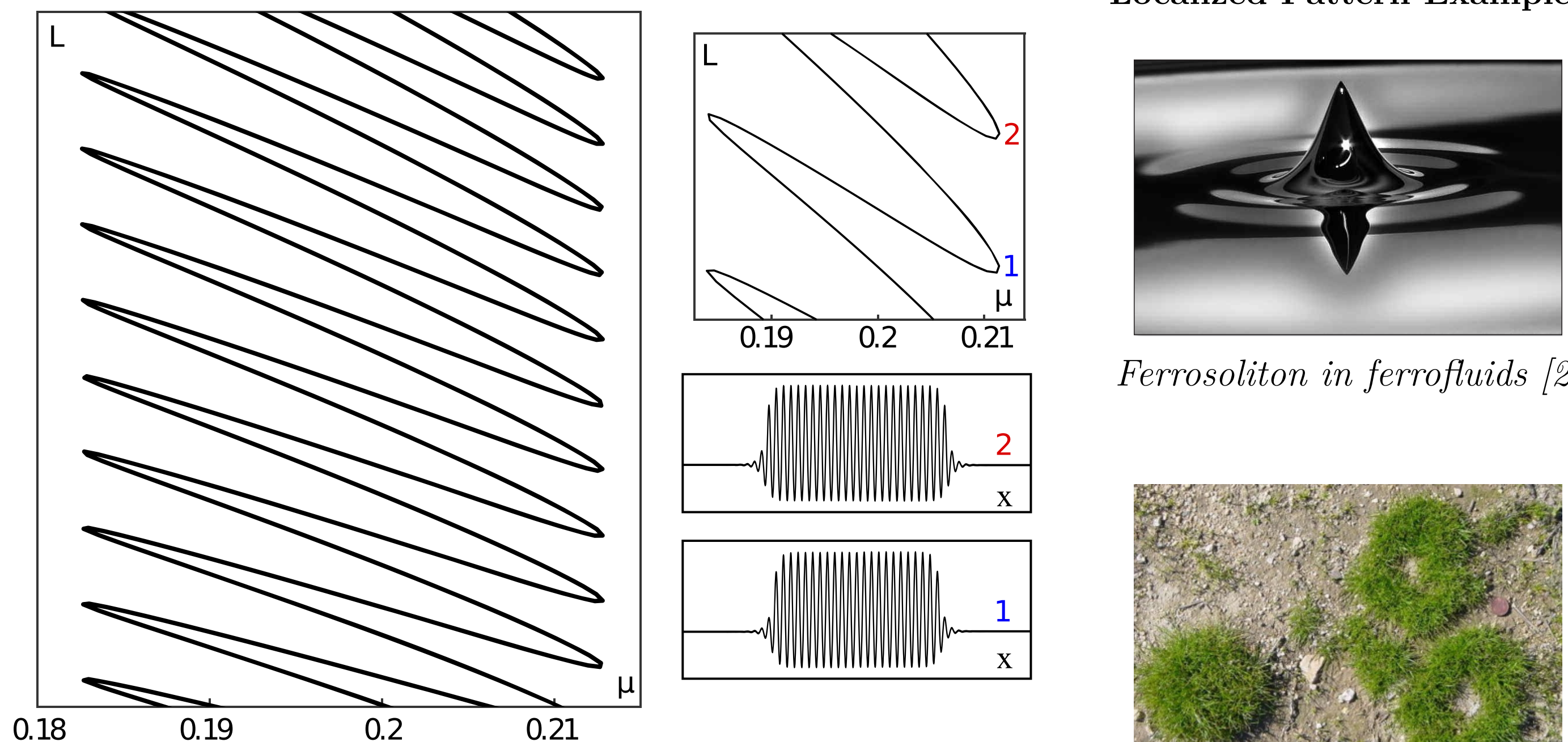


Localized Patterns

- Localized patterns, characterized by the coexistence of two distinct states, are observed in various natural and experimental phenomena. These patterns emerge as stationary solutions in systems governed by reaction-diffusion equations with bistable nonlinearity.
- Snaking bifurcations of patterns in spatially extended systems modeled by PDEs have been extensively observed and studied in previous research.

Localized Pattern Examples



Snaking existence curves and sample spatial profiles of localized patterns in Swift-Hohenberg equation. L is the length of the patterned region and μ is a parameter [1].

Ferrosoliton in ferrofluids [2]

Ring patterned grass [3]

Research Question

How does graph structure influence the connections of localized patterns?

- Our investigation begins with **ring graphs with symmetric coupling** where the structure can be varied through a controllable interaction length. We study the **bifurcations of localized patterns** in bistable reaction-diffusion systems imposed on these graphs.

The Lattice Dynamical System on Rings

The lattice dynamical system for a ring of $N \geq 3$ identical nodes:

$$\mathcal{F}(U, \mu, d)_n := \dot{u}_n = d(\Delta_m U)_n + f(u_n, \mu), \quad 1 \leq n \leq N, \quad U = (u_1, u_2, \dots, u_N) \in \mathbb{R}^N$$

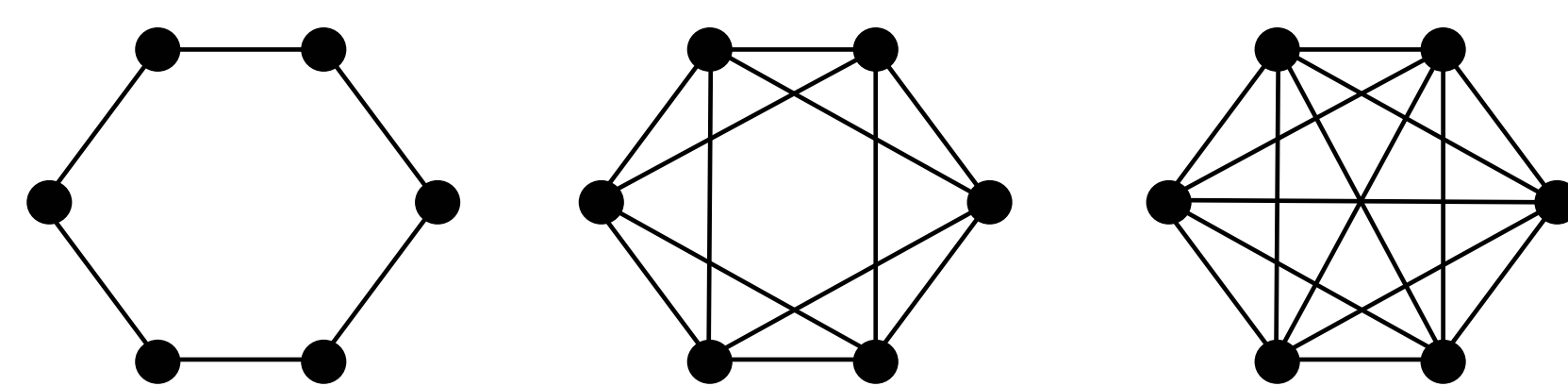
- All indices in the set $\{1, \dots, N\}$ modulo N
- d is coupling strength; $f(u, \mu)$ is a bistable nonlinearity
- Δ_m denotes the symmetric m -nearest-neighbour connections:

$$(\Delta_m U)_n = \begin{cases} -(2m+1)u_n + \sum_{j=-m}^m u_{n+j}, & 1 \leq m < \lfloor \frac{N}{2} \rfloor \\ -Nu_n + \sum_{j=1}^N u_j, & m = \lfloor \frac{N}{2} \rfloor \end{cases}$$

where the case $m = \lfloor \frac{N}{2} \rfloor$ is all-to-all coupling

Illustration of connections:

(From left to right) nearest neighbour, almost all-to-all, and all-to-all coupling

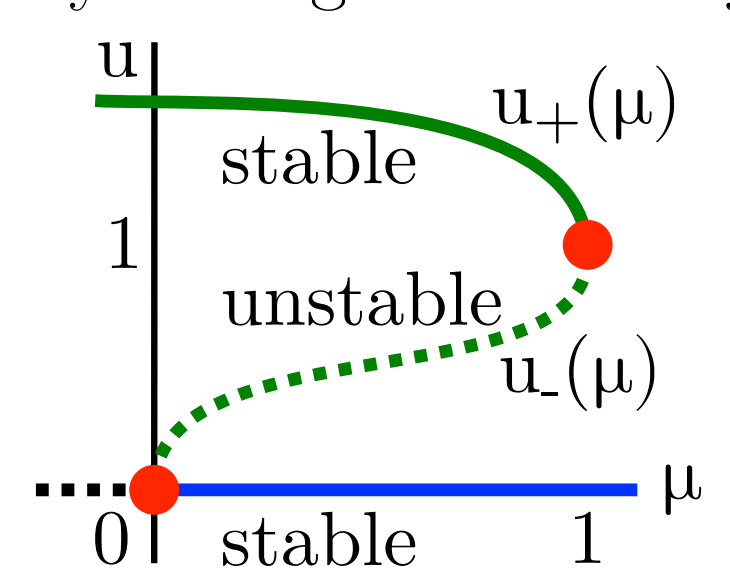


Approach: We use bifurcation theory to analyze for the regime $0 < d \ll 1$ (near the anti-continuum limit $d = 0$ when the system is uncoupled) and verify findings numerically.

For demonstration, we use the cubic-quintic nonlinearity:

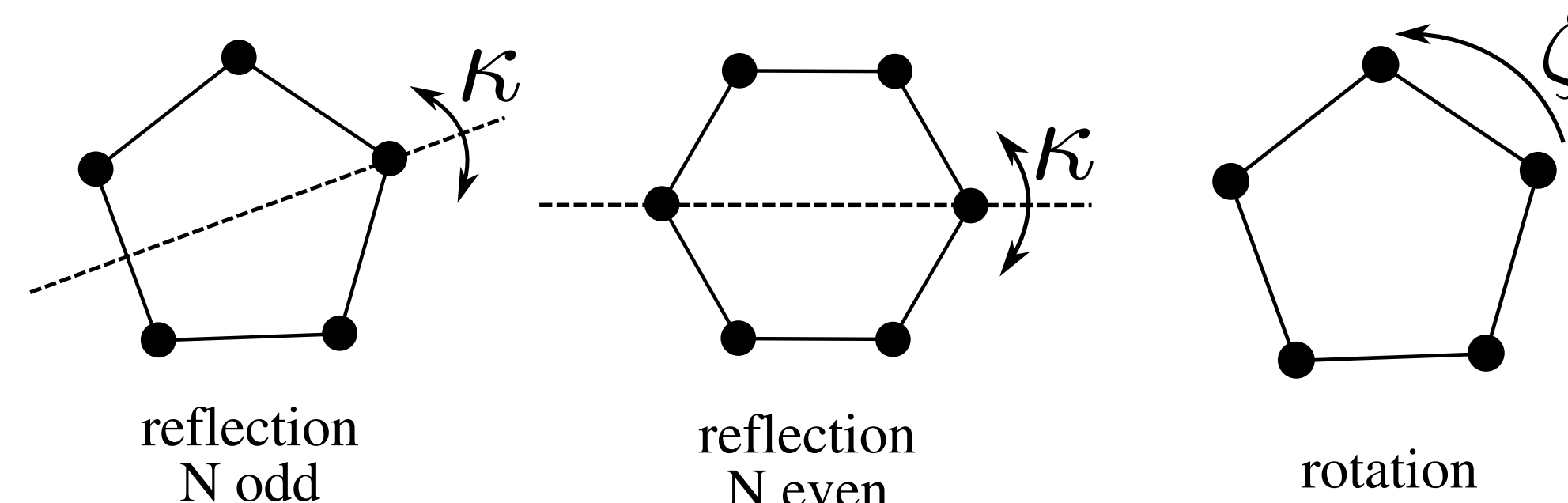
$$f(u, \mu) = -\mu u + 2u^3 - u^5.$$

The plot to the right illustrates the zero set $\{(u, \mu) : f(u, \mu) = 0\}$ of a typical bistable nonlinearity $f(u, \mu)$.



Group Actions:

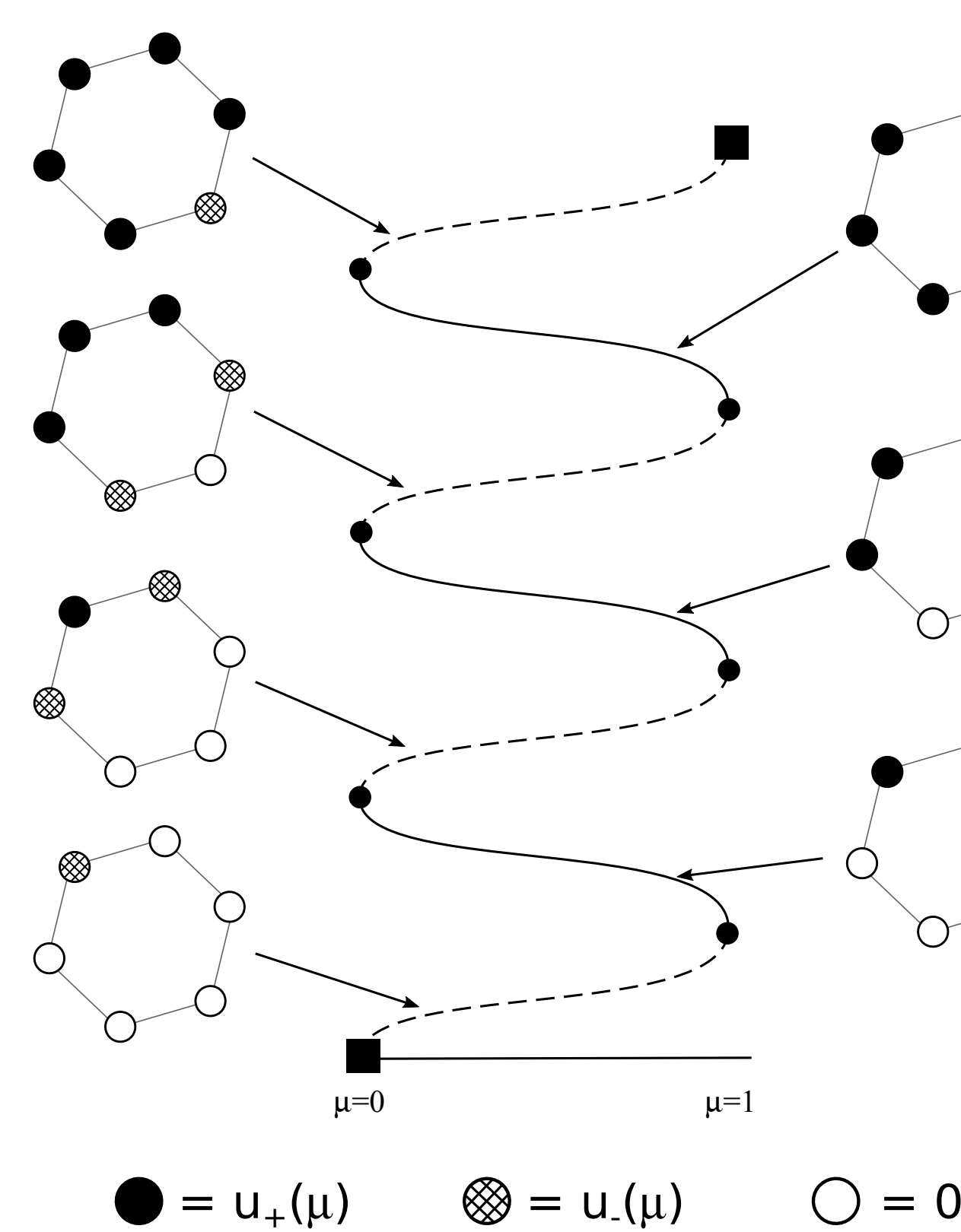
The ring system exhibits dihedral symmetry D_N .



Snaking Bifurcations

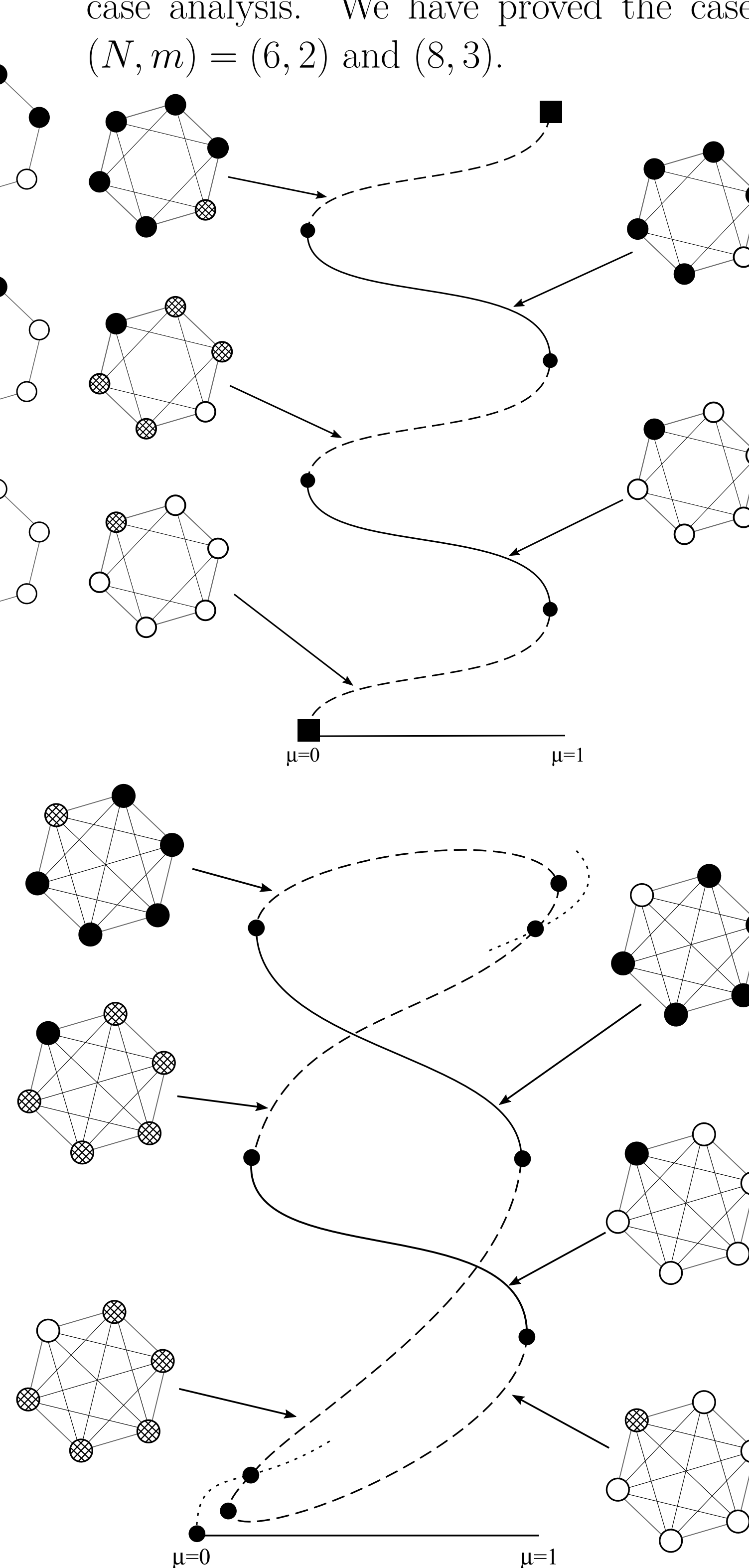
Sparse Coupling

With nearest ($m = 1$) and next-nearest ($m = 2$) neighbour couplings, snaking occurs with approximately N saddle nodes.



Almost All-to-all Coupling

When N is even and $m = \frac{N}{2} - 1$, the bifurcation diagram becomes different due to the new symmetries and requires case-by-case analysis. We have proved the cases $(N, m) = (6, 2)$ and $(8, 3)$.



All-to-all Coupling

In case of all-to-all coupling (i.e., a complete graph), the patterned solutions are connected through figure-eight-shaped branches with 6 saddle nodes, regardless of the number of nodes, N .

The dotted curve in the figure represents the branch of S_N -symmetric equilibria, where all nodes are equal to the unstable state $u_-(\mu)$, illustrating that the patterned solution branches originate and terminate at homogeneous patterns.

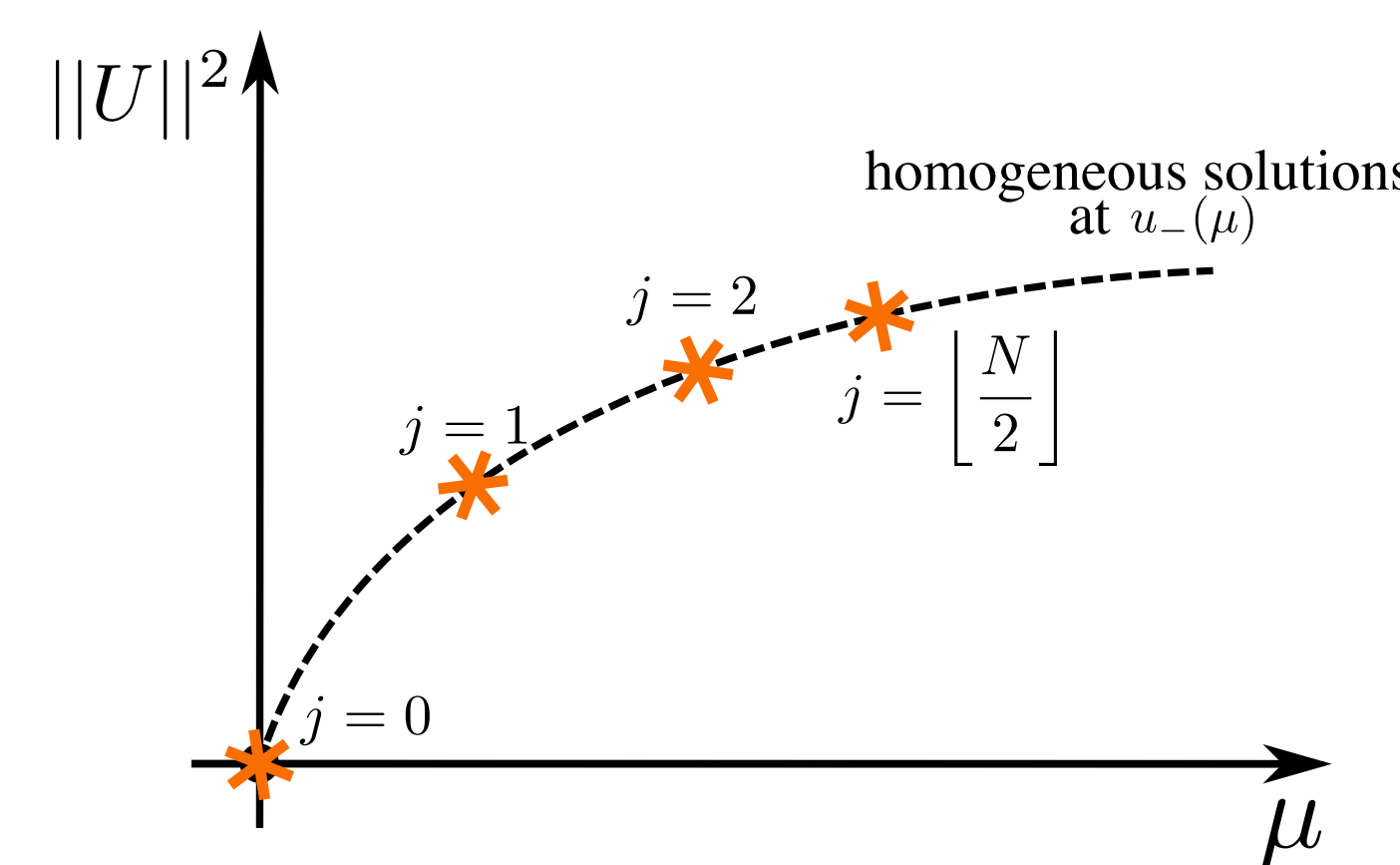
Homogeneous Solutions and Dihedral Group Bifurcation

Consider $N \geq 5$ and $1 \leq m \leq \lfloor \frac{N}{2} \rfloor - 1$. Let $\mathcal{L}(U, \mu, d) := D_U \mathcal{F}(U, \mu, d)$. We study bifurcations near $(u_n, \mu) = (0, 0)$ and $(u_n, \mu) = (1, 1)$.

- Along the branch of homogeneous solutions, $u_-(\mu)$, there are $\lfloor \frac{N}{2} \rfloor + 1$ bifurcation points, denoted by $(U_*^{(j)}(d), \mu_*^{(j)}(d), d)$, where $j = 0, 1, \dots, \lfloor \frac{N}{2} \rfloor$.

The eigenspaces correspond to those in the eigenproblem of the Laplacian operator, Δ_m .

- Focusing on $j = 1$ (a case verified against numerical simulations) and assuming $\dim \ker \mathcal{L}(U_*^{(1)}(d), \mu_*^{(1)}(d), d) = 2$ (validity proved for $m = 1$ when $N \geq 5$ and $m = 2$ when $N \geq 7$), we have

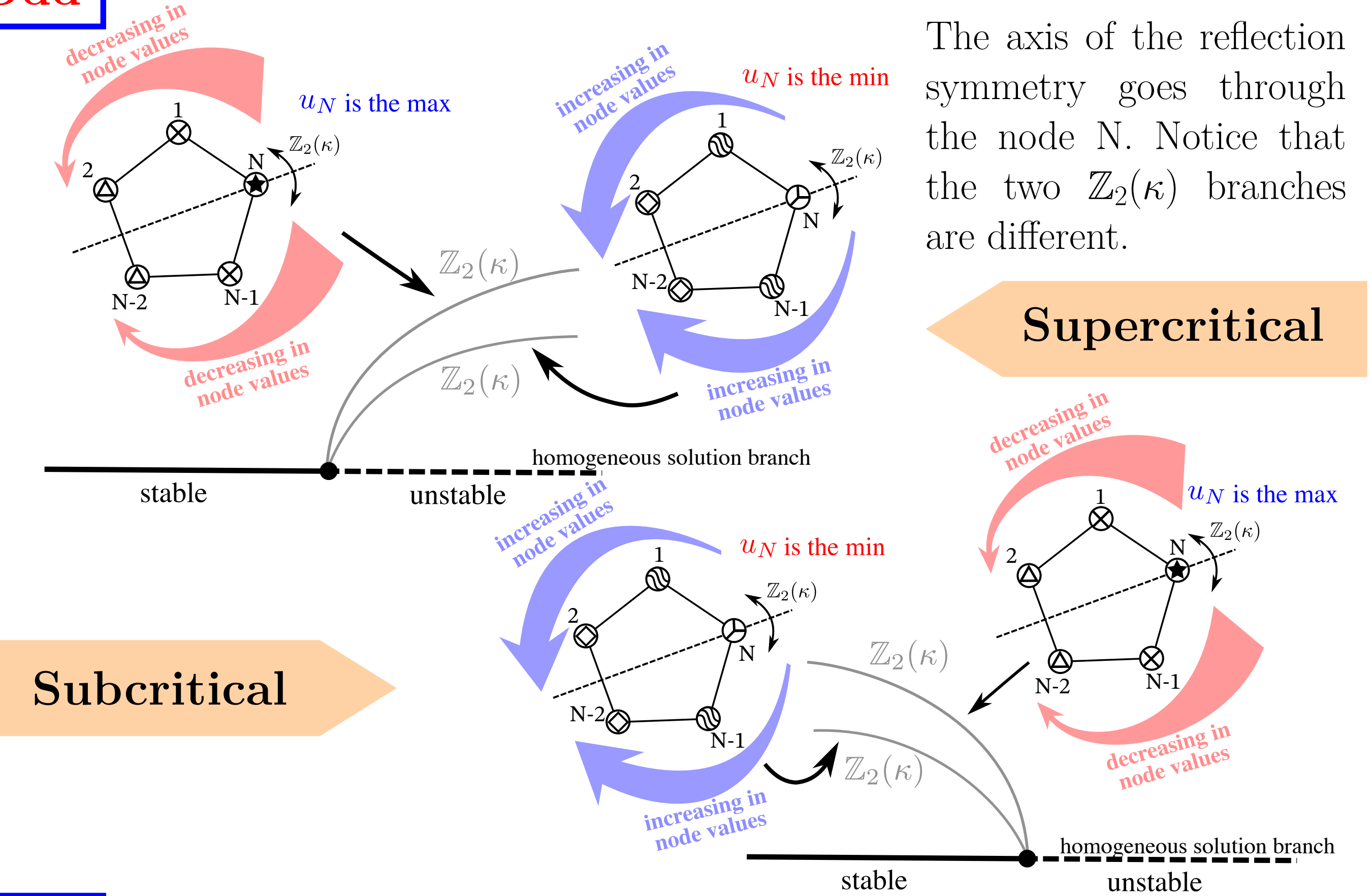


- proved the existence of patterned solution branches bifurcating from the $j = 1$ bifurcation point using bifurcation theory for D_N symmetry [4];
- derived formulae that determine the criticality for any N, m ;
- analyzed the $N \rightarrow \infty$ case and analytically computed the $\frac{m}{N}$ ratio at which the criticality transitions from supercritical to subcritical.

Bifurcation of Patterns from Homogeneous Branches

Two distinct branches bifurcate from the homogeneous solution branch $(U_*^{(1)}(d), \mu_*^{(1)}(d), d)$. Note: Patterns in the group orbit are also solutions on the same branch.

N Odd Patterns along the two branches have isotropy subgroup $\mathbb{Z}_2(\kappa)$.

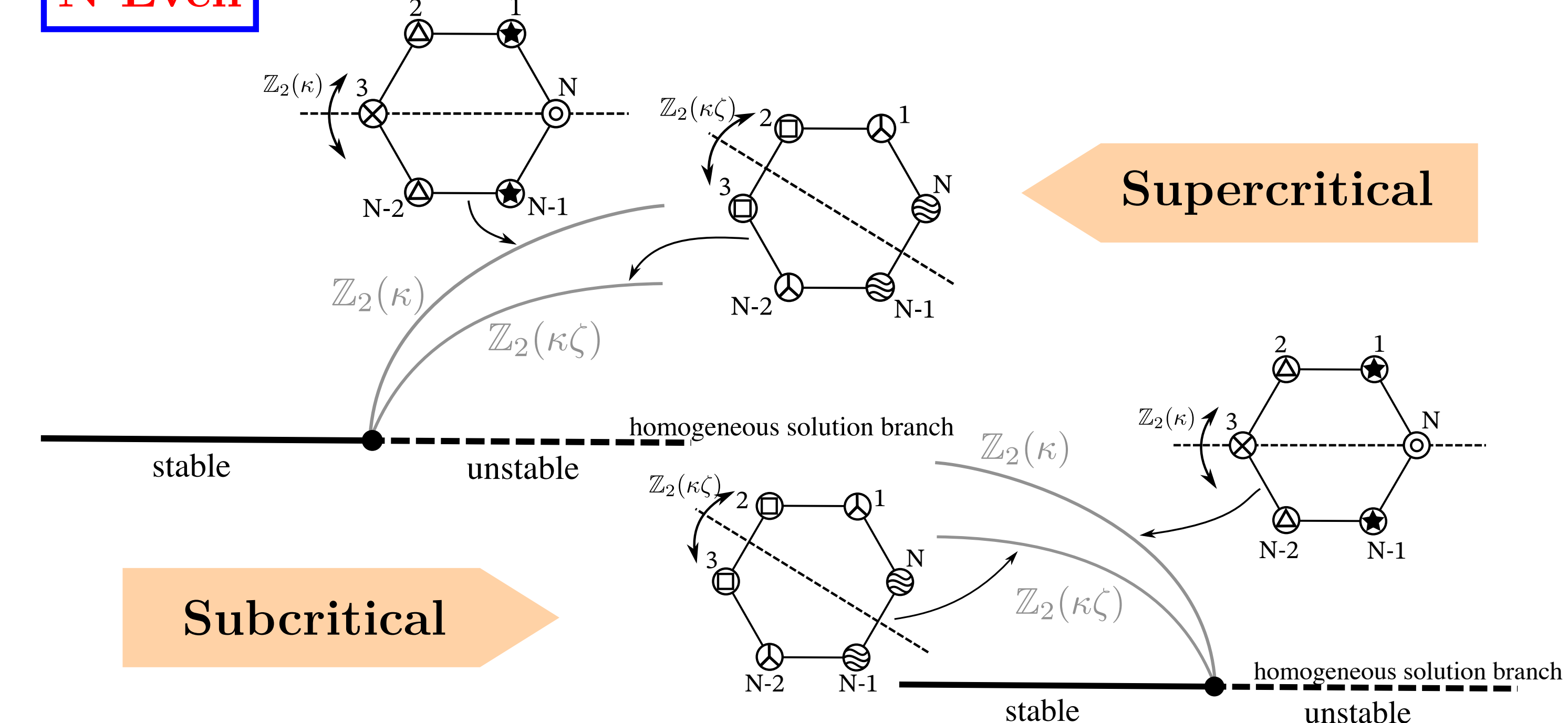


The axis of the reflection symmetry goes through the node N . Notice that the two $\mathbb{Z}_2(\kappa)$ branches are different.

Supercritical

Subcritical

N Even Patterns along the two branches have isotropy subgroup $\mathbb{Z}_2(\kappa)$ and $\mathbb{Z}_2(\kappa\zeta)$.



Supercritical

Subcritical

Conclusions

- Sparse coupling** exhibits **snaking** bifurcations while **all-to-all** coupling results in **closed figure-eight** curves. Snaking also occurs in **almost all-to-all** coupling but is more complex to study due to additional symmetries.
- The snaking curves of patterned solutions **bifurcate from homogeneous solutions**, and their criticalities can be determined analytically.

Future Work

- Generalize our results (extend to higher m , study the bifurcations at other j values, explore almost all-to-all coupling, etc)
- Investigate snaking and branch structures on more general finite graphs through computational studies [5]

References

- J.J. Bramburger and B. Sandstede. Localized patterns in planar bistable weakly coupled lattice systems. *Nonlinearity* **33**, (2020) 3500-3525.
- R. Richter. Magnetic liquid mountains. *Europhys. News* **42**(3), (2011) 17-19.
- E. Sheffer, H. Yizhaq, E. Gilad, M. Shachak and E. Meron. Why do plants in resource-deprived environments form rings? *Ecol. Complex.* **4**(4), (2007) 192-200.
- M. Golubitsky, I. Stewart and D.G. Schaeffer. *Singularities and groups in bifurcation theory. Vol. II, volume 69 of Applied Mathematical Sciences*. Springer-Verlag, New York (1988).
- N. McCullen and T. Wagenknecht. Pattern formation on networks: From localized activity to Turing patterns. *Sci. Rep.-UK* **6**, (2016) 27307.
- M. Tian, J. J. Bramburger and B. Sandstede. Snaking bifurcations of localized patterns on ring lattices. *IMA J. Appl. Math.* **86**(5), (2021)1112-1140.
- M. Tian. Patterns in network dynamics (2024). Applied Mathematics Theses and Dissertations. Brown Digital Repository. Brown University Library. <https://repository.library.brown.edu/studio/item/bdr:nfjwmw9c/>